$\alpha = 152$ MeV gives too small a $\mu^- + \mathrm{He}^3 \rightarrow \mathrm{H}^3 + \nu$ capture rate since it corresponds to too large a mean-square radius. (Raising α by about 20 MeV to obtain the correct radius would spoil the agreement with the photodisintegration data.[3])

We note that the previous calculations[6,15] of the $\mu^- + \mathrm{He}^3 \rightarrow \mathrm{H}^3 + \nu$, which have ranged from 1.40×10^3 to 1.66×10^3 sec^{-1}, differ primarily in the assumed nuclear wave function. The capture rate essentially depends only on the nuclear wave function through the mean-square radius, and the measurements of the capture rate lead to a radius of 1.6 to 1.7 F which is in agreement with values found by Hofstadter and collaborators in elastic e-He^3 and e-H^3 scattering.[16]

Finally we observe that *the class II axial-vector current enters in the muon-capture matrix element Eq. (1) in the same manner as the induced pseudoscalar term.* Consequently, unless the induced pseudoscalar contribution is accurately known, the presence of a small amount of the class II current cannot be detected.

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It is a pleasure to thank Professor J. S. Bell and Dr. T. A. Griffy for interesting discussions. We gratefully acknowledge the hospitality of the Physics Division of the Aspen Institute for Humanistic Studies where part of this work was performed.

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[15] A. Fujii, Phys. Rev. **118**, 870 (1960); C. Werntz, Nucl. Phys. **16**, 59 (1960); L. Wolfenstein, *Proceedings of the 1960 International Conference on High Energy Physics at Rochester* (University of Rochester, Rochester, 1960), p. 529; Bull. Am. Phys. Soc. **6**, 33 (1961); *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 821; A. F. Yano, Phys. Rev. Letters **12**, 110 (1964). See also A. Fujii and Y. Yamaguchi, Progr. Theoret. Phys. (Kyoto) **31**, 107 (1964) and W. Drechsler and B. Stech, Z. Physik **178**, 1 (1964).

[16] H. Collard, R. Hofstadter, E. B. Hughes, A. Johansson, M. R. Yearian, R. B. Day, and R. T. Wagner, "Proceedings of the 1964 International Conference on High Energy Physics at Dubna" (to be published). The radius is actually ambiguous in that the charge and magnetic radii differ.

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Theory of Hidden Variables

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It is shown that the stationary states of the nonrelativistic Schrödinger's equation are just the stationary states of a classical-mechanical system which is subject to random submicroscopic fluctuations of position. The proof covers the case (1) of a single particle moving in a potential, and (2) of two particles interacting through a potential $V(x_1 - x_2)$. The results can be easily generalized to the case of n interacting particles.

INTRODUCTION

In his theory of hidden variables in quantum mechanics, Bohm[1] has suggested that the uncertainty expressed by

$$(\Delta p)(\Delta q) \geq h \tag{1}$$

might be due to the presence of some random submicroscopic fluctuations which would introduce uncertainty into the otherwise classical equations of motion.

PART I

Let us then consider a function $\rho(x,t)$ such that

$$\int d^3x\, \rho(x,t) = 1 \,. \tag{2}$$

$\rho(x,t)$ is to be viewed either as the probability of finding the particle at the point x at time t, or as a function such that $m\rho(x,t)$ is the mass density of a continuous distribution of matter of total mass m. The two points of view will be interchangeable throughout the paper.

The particles are subject to random fluctuations, so in general there exists no velocity (the paths of the particles may be discontinuous). However, let us assume that if the particle was at x at time t, then at time $t+dt$ it will have a probability $w(t,x,dt,dx)$ of being found at the point $x+dx$. Since w is a probability distribution we have

$$\int w(t,x,dt,dx)\, d^3(dx) = 1 \,.$$

Then we define the velocity at x at time t by

$$v(x,t) = \lim_{dt \to 0} \left(\frac{1}{dt}\right) \int (dx)\, w(t,x,dt,dx)\, d^3(dx) \,.$$

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[1] D. Bohm, in *Quantum Theory, Radiation and High Energy Physics*, edited by D. Bates (Academic Press Inc., New York, 1962), Vol. III, p. 345.

Now let

$$W(t,x,dt,dx)=w(t,\,x,\,dt,\,dx+v(x,t)dt).$$

Then W is the probability that if the particle was at x at time t, at time $t+dt$ it will be at $x+v(x,t)dt+dx$. Also the above implies that

$$\lim_{dt\to 0}\left(\frac{1}{dt}\right)\int (dx)W(t,x,dt,dx)d^3(dx)=0.$$

We assume that the particle follows a path which is the superposition of the classical continuous path [given by $v(x,t)$] and a random Brownian motion fluctuation (which is independent of position and time). Therefore we require

$$W(t,x,dt,dx)=W(dt,dx)$$

and

$$\int (dx)^2W(dt,dx)d^3(dx)=Ddt, \qquad (3)$$

where D is an arbitrary constant. Now consider some time interval $\Delta t=Ndt$ and let $N\to\infty$ as $dt\to 0$ in such a way that Δt remains fixed. Then, if we let

$$\Delta x=\sum_{i=1}^{N}dx_i,$$

according to the central limit theorem of probability theory,[2] as $dt\to 0$, we have

$$W(\Delta t,\Delta x)=(2\pi D\Delta t)^{-3/2}\exp[-(\Delta x)^2/2D\Delta t]. \qquad (4)$$

Furthermore, we may still choose Δt as small as we like.[3]

Now the total displacement $\mathbf{\delta x}$ during the time interval Δt is $\mathbf{\delta x}=\mathbf{v}(x,t)\Delta t+\mathbf{\Delta x}$. Therefore the probability of going from x at time t to $x+\delta x$ at time $t+\Delta t$ is just

$$P(\delta x,\Delta t,x,t)=\frac{1}{(2\pi D\Delta t)^{3/2}}\exp(-(\delta x-v\Delta t)^2/2D\Delta t), \qquad (5)$$

where $v=v(x,t)$. This implies that

$$\rho(x,\,t+\Delta t)=\int\rho(x-\delta x,\,t)P(\delta x,\,\Delta t,\,x-\delta x,\,t)d^3(\delta x). \qquad (6)$$

In the limit as $\Delta t\to 0$ we may write

$$\rho(x-\delta x,\,t)=\rho(x,t)-\sum\delta x_j\nabla_j\rho(x,t)+\tfrac{1}{2}\sum\sum\delta x_i\delta x_j\nabla_i\nabla_j\rho(x,t) \qquad (7)$$

and

$$P(x-\delta x,\,t)=P(x,t)-\sum\delta x_j\nabla_jP(x,t)+\tfrac{1}{2}\sum\sum\delta x_i\delta x_j\nabla_i\nabla_jP(x,t). \qquad (8)$$

Retaining only terms of first order in Δt, we have

$$\rho(x,\,t+\Delta t)=\rho(x,t)-\Delta t\sum\nabla_j(\rho v_j)+\Delta t\frac{D}{2}\nabla^2\rho, \qquad (9)$$

which implies that

$$\frac{\partial\rho}{\partial t}=-\nabla_j(\rho v_j)+\tfrac{1}{2}D\nabla^2\rho. \qquad (10)$$

This implies that

$$\frac{\partial\rho}{\partial t}+\nabla_j\left[\rho\left(v_j-\tfrac{1}{2}D\frac{\nabla_j\rho}{\rho}\right)\right]=0. \qquad (11)$$

The quantity $(D/2)(\nabla\rho/\rho)$ is just the diffusion velocity.

Let $V(x)$ be the potential field in which the particle moves. Then we assume

$$v_j(x,\,t+\Delta t)=\frac{1}{N}\int\left[v_j(x-\delta x,\,t)-\Delta t\nabla_j\frac{V(x-\delta x)}{m}\right]\times\rho(x-\delta x,\,t)P(\delta x,\,\Delta t,\,x-\delta x,\,t)d^3(\delta x), \qquad (12)$$

where N is the normalization constant

$$N=\int\rho(x-\delta x,\,t)P(\delta x,\,\Delta t,\,x-\delta x,\,t)d^3(\delta x)$$
$$=\rho(x,t)-\Delta t\nabla_j(\rho v_j)+\Delta t(\tfrac{1}{2}D)\nabla^2\rho. \qquad (13)$$

The factor of $[v_j(x-\delta x,\,t)-(\Delta t)(1/m)\nabla_jV(x-\delta x)]$ appears because the particles had velocity $v_j(x-\delta x,\,t)$ at $x-\delta x$ at time t and received an additional velocity increment $[-(\Delta t)(1/m)\nabla_jV(x-\delta x)]$ due to the force $[-\nabla_jV]$ on the particles. The factor $\rho(x-\delta x,\,t)P(\delta x,\,\Delta t,\,x-\delta x,\,t)$ appears because the total number of particles arriving at x from $x-\delta x$ is equal to the number of particles at $x-\delta x$ multiplied by the probability of a particle going from $x-\delta x$ to x in time Δt. We then average over all δx to obtain the mean velocity $v(x,t+\Delta t)$.

Expanding v, ρ, P, and V in a Taylor series, integrating and retaining terms of only first order in Δt, we get

$$\left(v_j(x,t)+\Delta t\frac{\partial v_j}{\partial t}\right)(\rho-\Delta t\nabla_j(\rho v_j)+(\tfrac{1}{2}D)\nabla^2\rho)$$
$$=\rho v_j-\Delta t(\rho v_i)\nabla_iv_j-\Delta tv_j\nabla_i(\rho v_i)+\Delta t(\tfrac{1}{2}D)\nabla^2(\rho v_j)-\rho\Delta t\nabla_j\frac{V(x)}{m}, \qquad (14)$$

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[2] See, for example, N. Wax, *Selected Papers on Noise and Stochastic Processes* (Dover Publications, New York, 1954), pp. 17 and 18. Pages 1–44 of this book are an excellent introduction to the type of method used in this paper.

[3] The limiting process here is clearly suspect; however, I feel that it gives more insight into the nature of the assumptions being made than if I just arbitrarily defined

$$W(\Delta t,\Delta x)=\frac{1}{(2\pi D\Delta t)^{3/2}}\exp\left(\frac{-(\Delta x)^2}{2D\Delta t}\right).$$

Furthermore, these assumptions are just the standard ones of the theory of Brownian motion.

which implies that

$$m\left(\frac{\partial v_j}{\partial t}+v_i\nabla_i v_j\right)=m\frac{dv_j}{dt}=-\nabla_j V(x)$$

$$+(\tfrac{1}{2}D)m\left(\frac{\nabla^2(\rho v_j)}{\rho}-v_j\frac{\nabla^2\rho}{\rho}\right). \quad (15)$$

I shall concentrate on the stationary state solutions of the above equations.[4] If we are to have a stationary state, then the diffusion velocity must just counterbalance the mean particle velocity. That is, $v(x,t)=(D/2)(\nabla\rho/\rho)$, implies [by Eq. (10)]

$$\frac{\partial\rho}{\partial t}=0\rightarrow\frac{\partial v_j}{\partial t}=0, \quad (16)$$

putting these into Eq. (15) gives

$$m\left(\frac{D}{2}\right)^2\left[\frac{\nabla_i\rho}{\rho}\nabla_i\right]\frac{\nabla_j\rho}{\rho}$$

$$=-\nabla_j V+\left(\frac{D}{2}\right)^2 m\left(\frac{\nabla^2\nabla_j\rho}{\rho}-\frac{\nabla_j\rho}{\rho}\frac{\nabla^2\rho}{\rho}\right)$$

$$=-\nabla_j V(x)+\left(\frac{D}{2}\right)^2 m\nabla_j\left(\frac{\nabla^2\rho}{\rho}\right). \quad (17)$$

Now

$$\frac{\nabla_j\rho}{\rho}=\nabla_j\ln\rho\rightarrow\epsilon_{ijk}\nabla_j\left(\frac{\nabla_k\rho}{\rho}\right)=0 \quad (18)$$

implies that

$$\nabla_j\left[-D^2\frac{m}{2}\left(\frac{\nabla^2\rho}{2\rho}-\left(\frac{\nabla\rho}{2\rho}\right)^2\right)+V(x)\right]=0. \quad (19)$$

However

$$\frac{\nabla^2\rho}{2\rho}-\left(\frac{\nabla\rho}{2\rho}\right)^2=\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}, \quad (20)$$

and a function whose gradient is everywhere zero is a constant so

$$-\frac{(Dm)^2}{2m}\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}+V(x)=\text{constant}. \quad (21)$$

To evaluate the constant we observe that the total energy of the system is given by

$$E=\int\rho(x,t)[\tfrac{1}{2}mv^2+V(x)]d^3x. \quad (22)$$

For our case $v(x,t)=(D/2)(\nabla\rho/\rho)$, which implies that

$$E=\int\rho(x,t)\left(\left(\frac{D}{2}\right)^2\left(\frac{m}{2}\right)\left(\frac{\nabla\rho}{\rho}\right)^2+V(x)\right)d^3x. \quad (23)$$

If $\rho\rightarrow 0$ as $x\rightarrow\infty$ then

$$E=\int\rho(x,t)\left\{-\frac{(Dm)^2}{2m}\left[\frac{\nabla^2\rho}{2\rho}-\left(\frac{\nabla\rho}{2\rho}\right)^2\right]+V(x)\right\}d^3x$$

$$=\int\rho(x,t)(\text{constant})d^3x=\text{constant}. \quad (24)$$

Finally then we have

$$-\frac{(Dm)^2}{2m}\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}+V(x)=E, \quad (25)$$

which is just Schrödinger's equation for the stationary state,

$$\psi=(\sqrt{\rho})\exp(-i(E/\hbar)t), \quad (26)$$

all we need do is put $D=\hbar/m$.

It may seem strange that $v(x,t)$ is not zero [i.e., $v(x,t)=(D/2)(\nabla\rho/\rho)\neq 0$] while the solution is supposed to be a stationary one. It must be remembered that the total path is the sum of the $v(x,t)$ part and the random fluctuation part. For stationary solutions the displacements due to the random fluctuations, on the average, just cancel the displacements due to the mean velocity.

PART II

Now we shall consider the problem for two interacting particles since this is the problem of real physical interest. Let $\rho(x_1,x_2,t)$ be the probability of finding the first particle at x_1 and the second particle at x_2 at time t. Let there be a potential force $V(x_1-x_2)$ operating between the two particles, and let $v_1(x_1,x_2,t)$ and $v_2(x_1,x_2,t)$ be their respective velocities. There is no good reason why the velocity of the one particle should be statistically independent of the position of the other, so we write $v_1(x_1,x_2,t)$ rather than $v_1(x_1,t)$.

Then

$$\int d^3x_1 d^3x_2\,\rho(x_1,x_2,t)=1 \quad (27)$$

and the random displacements Δx_1 and Δx_2 of the two particles are governed by:

$$W^1(\Delta t,\Delta x_1)=\frac{1}{(2\pi\hbar\Delta t/m_1)^{3/2}}\exp\left(-\frac{m_1(\Delta x_1)^2}{2\Delta t\hbar}\right), \quad (28)$$

$$W^2(\Delta t,\Delta x_2)=\frac{1}{(2\pi\hbar\Delta t/m_2)^{3/2}}\exp\left(-\frac{m_2(\Delta x_2)^2}{2\Delta t\hbar}\right), \quad (29)$$

since in Part I we showed that $D=\hbar/m$, and m_1 is the mass of the first particle and m_2 the mass of the second.

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[4] I have, as yet, had no success in showing that the nonstationary solutions of Eqs. (11) and (15) are just the nonstationary solutions of Schrödinger's equation.

Now let $r=x_1-x_2$ and $R=(m_1x_1+m_2x_2)/(m_1+m_2)$, then

$$W(\Delta t,\Delta r)=\frac{\partial(x_2)}{\partial(r)}\int W^1(\Delta t,\Delta x)W^2(\Delta t,\, \Delta x-\Delta r)d^3(\Delta x), \tag{30}$$

where $\partial(x_2)/\partial(r)$ is the Jacobian of x_2 with respect to r. This implies that

$$\begin{aligned} W(\Delta t\,\Delta r)&=\int\frac{d^3(\Delta x)}{[(2\pi\hbar\Delta t)^2/m_1m_2]^{3/2}}\exp\left(-\frac{[m_1(\Delta x)^2+m_2(\Delta x-\Delta r)^2]}{2\hbar\Delta t}\right)\\ &=\left[\frac{\exp(-\mu(\Delta r)^2/2\hbar\Delta t)}{(2\pi\hbar\Delta t/\mu)^{3/2}}\right]\int\frac{d^3(\Delta x)}{[2\pi\hbar\Delta t/(m_1+m_2)]^{3/2}}\exp\left(-\frac{(m_1+m_2)\{\Delta x-[m_2/(m_1+m_2)]\Delta r\}^2}{2\hbar\Delta t}\right), \end{aligned}\tag{31}$$

or finally

$$W(\Delta t\,\Delta r)=\frac{1}{(2\pi\hbar\Delta t/\mu)^{3/2}}\exp\left(-\frac{\mu(\Delta r)^2}{2\hbar\Delta t}\right), \tag{32}$$

where $\mu=(m_1m_2)/(m_1+m_2)=$ reduced mass. Similarly

$$W(\Delta t,\Delta R)=\frac{\partial(x_2)}{\partial(R)}\int W^1(\Delta t\,\Delta x)W^2(\Delta t,\,(M/m_2)\Delta R-(m_1/m_2)\Delta x)d^3(\Delta x). \tag{33}$$

This implies that

$$\begin{aligned} W(\Delta t,\Delta R)&=\left(\frac{M}{m_2}\right)^3\int\frac{d^3(\Delta x)}{((2\pi\hbar\Delta t)^2/m_1m_2)^{3/2}}\exp\left(-\left[m_1(\Delta x)^2+m_2\left(\frac{M}{m_2}\Delta R-\frac{m_1}{m_2}\Delta x\right)^2\right]\Big/2\hbar\Delta t\right)\\ &=\left[\frac{\exp(-M(\Delta R)^2/2\hbar\Delta t)}{(2\pi\hbar\Delta t/M)^{3/2}}\right]\int\frac{d^3(\Delta x)}{[2\pi\hbar\Delta t/(Mm_1/m_2)]^{3/2}}\exp\left(-\frac{(Mm_1/m_2)(\Delta x-\Delta R)^2}{2\hbar\Delta t}\right), \end{aligned}\tag{34}$$

or finally

$$W(\Delta t,\Delta R)=\frac{1}{(2\pi\hbar\Delta t/M)^{3/2}}\exp\left(-\frac{M(\Delta R)^2}{2\hbar\Delta t}\right), \tag{35}$$

where $M=m_1+m_2=$ total mass. Now let

$$\begin{aligned} S(r,R,t)&=(m_1v_1+m_2v_2)/M\\ c(r,R,t)&=v_1-v_2. \end{aligned}\tag{36}$$

Then the total change in position is given by

$$\begin{aligned} \delta R&=S\Delta t+\Delta R\\ \delta r&=c\Delta t+\Delta r. \end{aligned}\tag{37}$$

We can rewrite $\rho(x_1,x_2,t)$ as

$$\rho(r,R,t)=\rho(x_1,x_2,t)\frac{\partial(x_1,x_2)}{\partial(r,R)}\cdot \tag{38}$$

We have then as before:

$$P_R(\delta R,\Delta t,r,R,t)=\frac{1}{(2\pi\hbar\Delta t/M)^{3/2}}\exp\left(-\frac{M(\delta R-S\Delta t)^2}{2\hbar\Delta t}\right) \tag{39}$$

$$P_r(\delta r,\Delta t,r,R,t)=\frac{1}{(2\pi\hbar\Delta t/\mu)^{3/2}}\exp\left(-\frac{\mu(\delta r-c\Delta t)^2}{2\hbar\Delta t}\right), \tag{40}$$

which implies that

$$\begin{aligned} \rho(r,\,R,\,t+\Delta t)&=\int\rho(r-\delta r,\,R-\delta R,\,t)P_r(\delta r,\,\Delta t,\,r-\delta r,\,R-\delta R,\,t)P_R(\delta R,\,\Delta t,\,r-\delta r,\,R-\delta R,\,t)d^3\delta r d^3\delta R\\ &=\rho-\Delta t[\boldsymbol{\nabla}_{rj}(\rho c_j)+\boldsymbol{\nabla}_{Rj}(\rho S_j)]+\Delta t\frac{\hbar}{2}\left(\frac{\nabla_r{}^2\rho}{\mu}+\frac{\nabla_R{}^2\rho}{M}\right). \end{aligned}\tag{41}$$

This implies (as before) that

$$\rho(r,\, R,\, t+\Delta t)=\rho(r,R,t)-\Delta t[\boldsymbol{\nabla}_{rj}(\rho c_j)+\boldsymbol{\nabla}_{Rj}(\rho S_j)]+\Delta t\frac{\hbar}{2}\left(\frac{\nabla_r^2\rho}{\mu}+\frac{\nabla_R^2\rho}{M}\right), \tag{42}$$

which implies that

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}_{rj}\left[\rho\left(c_j-\frac{\hbar}{2\mu}\frac{\boldsymbol{\nabla}_{rj}\rho}{\rho}\right)\right]+\boldsymbol{\nabla}_{Rj}\left[\rho\left(S_j-\frac{\hbar}{2M}\frac{\boldsymbol{\nabla}_{Rj}\rho}{\rho}\right)\right]=0. \tag{43}$$

The potential $V(r)$ affects only c and not S. We have then

$$S(r,\, R,\, t+\Delta t)=\frac{1}{N}\int S(r-\delta r,\, R-\delta R,\, t)\rho(r-\delta r,\, R-\delta R,\, t)P_rP_Rd^3\delta rd^3\delta R, \tag{44}$$

$$N=\int\rho(r-\delta r,\, R-\delta R,\, t)P_rP_Rd^3\delta rd^3\delta R$$

$$=\rho-\Delta t[\boldsymbol{\nabla}_{rj}(\rho c_j)+\boldsymbol{\nabla}_{Rj}(\rho S_j)]+\Delta t\frac{\hbar}{2}\left[\frac{\nabla_r^2\rho}{\mu}+\frac{\nabla_R^2\rho}{M}\right], \tag{45}$$

and

$$c(r,\, R,\, t+\Delta t)=\frac{1}{N}\int\left[c(r-\delta r,\, R-\delta R,\, t)-\Delta t\frac{\boldsymbol{\nabla}_rV(r-\delta r)}{\mu}\right]\rho(r-\delta r,\, R-\delta R,\, t)P_rP_Rd^3\delta rd^3\delta R. \tag{46}$$

Expanding S, c, P_r, P_R, and $V(r)$ in Taylor series, integrating, and retaining terms only of first order in Δt, we obtain

$$M\left[\frac{\partial S_j}{\partial t}+(S_i\boldsymbol{\nabla}_{Ri})S_j+(c_i\boldsymbol{\nabla}_{ri})S_j\right]=M\frac{dS_j}{dt}=M\left\{\frac{\hbar}{2\mu}\left(\frac{\nabla_r^2(\rho S_j)}{\rho}-S_j\frac{\nabla_r^2\rho}{\rho}\right)+\frac{\hbar}{2M}\left(\frac{\nabla_R^2(\rho S_j)}{\rho}-S_j\frac{\nabla_R^2\rho}{\rho}\right)\right\}, \tag{47}$$

and

$$\mu\left(\frac{\partial c_j}{\partial t}+(S_i\boldsymbol{\nabla}_{Ri})c_j+(c_i\boldsymbol{\nabla}_{ri})c_j\right)=\mu\frac{dc_j}{dt}=-\boldsymbol{\nabla}_{rj}V(r)+\mu\left\{\frac{\hbar}{2\mu}\left(\frac{\nabla_r^2(\rho c_j)}{\rho}-c_j\frac{\nabla_r^2\rho}{\rho}\right)+\frac{\hbar}{2M}\left(\frac{\nabla_R^2(\rho c_j)}{\rho}-c_j\frac{\nabla_R^2\rho}{\rho}\right)\right\}. \tag{48}$$

Again we shall concentrate on the stationary state solutions. As before we put $c=h/2\mu(\boldsymbol{\nabla}_r\rho/\rho)$ and $S=h/2M\times(\boldsymbol{\nabla}_R\rho/\rho)$ which gives by Eq. (43)

$$\partial\rho/\partial t=0. \tag{49}$$

and

$$\partial c_j/\partial t=\partial S_j/\partial t=0. \tag{50}$$

By analogy to the separation of variables in quantum mechanics, we assume that

$$\rho(r,R)=\rho_r(r)\rho_R(R), \tag{51}$$

implying that

$$c(r,R)=c(r) \tag{52}$$

and

$$S(r,R)=S(R). \tag{53}$$

Putting these into Eqs. (47) and (48) we get

$$-\frac{\hbar^2}{2M}\frac{\nabla_R^2(\sqrt{\rho_R})}{\sqrt{\rho_R}}=E_R \tag{54}$$

and

$$-\frac{\hbar^2}{2\mu}\frac{\nabla_r^2(\sqrt{\rho_r})}{\sqrt{\rho_r}}=E_r-V(r), \tag{55}$$

where the identification of the integration constants with the energies has been made by the same method as in Part I.

If we let

$$\psi_R = (\sqrt{\rho_R}) \exp[-i(E_R/\hbar)t] \tag{56}$$

and

$$\psi_r = (\sqrt{\rho_r}) \exp[-i(E_r/\hbar)t], \tag{57}$$

we have just Schrödinger's equation for a two-particle system.

$$E_R\psi_R = -\frac{\hbar^2}{2M}\nabla_R{}^2\psi_R \tag{58}$$

and

$$E_r\psi_r = -\frac{\hbar^2}{2\mu}\nabla_r{}^2\psi_r + V(r)\psi_r. \tag{59}$$

PART III

At this point some physical interpretation of the preceding equations in in order. I think that they are best understood by relating them to the theory of stationary Markov chains.[5] The state of our system is described by giving its position x and its velocity v. Our knowledge of the system at any time t is described by a probability function $P(x,v,t)$, and the two-step transition probability was shown to be

$$w(x,\, v,\, x+\delta x,\, v+\delta v,\, \Delta t) = \frac{1}{(2\pi D\Delta t)^{3/2}} \exp\left(-\frac{(\delta x - v\Delta t)^2}{2D\Delta t}\right)\delta^3\left(\delta v + \Delta t\frac{\nabla V(x)}{m}\right), \tag{60}$$

where $\delta^3(x)$ is Dirac's delta function. We also showed that stationary probability distributions were of the form

$$P(x,v) = \rho(x)\delta^3\left(v - \frac{D}{2}\frac{\nabla\rho}{\rho}\right). \tag{61}$$

Now if $P(x,v,t) = P(x,v)$ is to be stationary it must satisfy

$$P(x,v) = \int w(x-\delta x,\, v-\delta v,\, x,\, v,\, t)P(x-\delta x,\, v-\delta v)d^3(\delta x)d^3(\delta v). \tag{62}$$

But, it is clear that $P(x,v)$ does not satisfy this condition for it could not possibly maintain a delta function velocity distribution. $P(x,v)$ is however stationary in the sense that if we let

$$Q(x,v) = \int w(x-\delta x,\, v-\delta v,\, x,\, v,\, t)P(x-\delta x,\, v-\delta v)d^3(\delta x)d^3(\delta v), \tag{63}$$

then

$$P(x,v) = \left[\int d^3v\, Q(x,v)\right]\delta^3\left[v - \frac{\int vQ(x,v)d^3v}{\int Q(x,v)d^3v}\right]. \tag{64}$$

To show that this is so we have from Eqs. (60), (61), and (64)

$$\begin{aligned} Q(x,v) &= \int \frac{d^3\delta x d^3\delta v}{(2\pi D\Delta t)^{3/2}}\rho(x-\delta x)\delta^3\left(\delta v + \Delta t\frac{\nabla V}{m}\right)\delta^3\left(v - \delta v - \frac{D}{2}\frac{\nabla\rho}{\rho}(x-\delta x)\right)\exp\left(-\frac{[\delta x - (v-\delta v)\Delta t]^2}{2D\Delta t}\right) \\ &= \int \frac{d^3\delta x}{(2\pi D\Delta t)^{3/2}}\rho(x-\delta x)\delta^3\left(v - \frac{D}{2}\frac{\nabla\rho}{\rho}(x-\delta x) + \Delta t\frac{\nabla V(x)}{m}\right)\exp\left(-\frac{[\delta x - \Delta t(D/2)(\nabla\rho/\rho)(x-\delta x)]^2}{2D\Delta t}\right), \end{aligned} \tag{65}$$

hence

$$\int Q(x,v)d^3v = \int \frac{d^3\delta x}{(2\pi D\Delta t)^{3/2}}\rho(x-\delta x)\exp\left(-\frac{[\delta x - \Delta t(D/2)(\nabla\rho/\rho)(x-\delta x)]^2}{2D\Delta t}\right) \tag{66}$$

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[5] See, for example, A. I. Khinchin, *Mathematical Foundations of Information Theory* (Dover Publications, New York, 1957).

and to order Δt Eq. (66) becomes

$$
\begin{aligned}
&=\rho(x)-\Delta t\frac{D}{2}\left(\frac{\boldsymbol{\nabla}_i\rho}{\rho}\boldsymbol{\nabla}_i\right)\rho-\Delta t\frac{D}{2}\rho\boldsymbol{\nabla}_i\left(\frac{\boldsymbol{\nabla}_i\rho}{\rho}\right)+\Delta t\frac{D}{2}\nabla^2\rho \\
&=\rho(x)-\Delta t\frac{D}{2}\nabla^2\rho+\Delta t\frac{D}{2}\nabla^2\rho=\rho(x).
\end{aligned}
\tag{67}
$$

Similarly

$$
\left(\int vQ(x,v)d^3v\right)\Big/\left(\int Q(x,v)d^3v\right)
$$

$$
=\frac{1}{\rho(x)}\int\frac{d^3\delta x}{(2\pi D\Delta t)^{3/2}}\left[\frac{D}{2}\boldsymbol{\nabla}\rho(x-\delta x)-\Delta t\rho(x-\delta x)\frac{\boldsymbol{\nabla}V(x)}{m}\right]\exp\left(-\frac{[\delta x-\Delta t(D/2)(\boldsymbol{\nabla}\rho/\rho)(x-\delta x)]^2}{2D\Delta t}\right)
\tag{68}
$$

and to order Δt Eq. (68) becomes

$$
\begin{aligned}
&=\frac{D}{2}\frac{\boldsymbol{\nabla}\rho}{\rho}+\Delta t\left\{-\frac{\boldsymbol{\nabla}V}{m}-\left(\frac{D}{2}\right)^2\left[\frac{\boldsymbol{\nabla}_i\rho}{\rho^2}\boldsymbol{\nabla}_i\right]\boldsymbol{\nabla}\rho-\left(\frac{D}{2}\right)^2\frac{\boldsymbol{\nabla}\rho}{\rho}\left[\nabla_i\left(\frac{\boldsymbol{\nabla}_i\rho}{\rho}\right)\right]+\left(\frac{D}{2}\right)^2\frac{(\nabla^2\boldsymbol{\nabla}\rho)}{\rho}\right\} \\
&=\frac{D}{2}\frac{\boldsymbol{\nabla}\rho}{\rho}+\Delta t\boldsymbol{\nabla}\left(-\frac{V(x)}{m}+\frac{D^2}{2}\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\right)=v=\frac{D}{2}\frac{\boldsymbol{\nabla}\rho}{\rho}
\end{aligned}
\tag{69}
$$

since

$$
-\frac{D^2}{2}m\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}+V(x)=0. \quad \text{(Q.E.D.)}
\tag{70}
$$

We have shown that Eqs. (60), (61), and (64) taken together imply that the square root of $\rho(x)$ satisfies Schrödinger's equation.

Hence the stationary solutions of Schrödinger's equation are just the stationary probability distributions of the motion of the system considered as a Markov chain.

One other observation is germane. R. P. Feynman, in his "Space-Time Approach to Non-Relativistic Quantum Mechanics,"[6] shows that most of the contribution to $\psi(x_{k+1},t+\Delta t)$ comes from x_k such that $|x_{k+1}-x_k|^2\sim\hbar\Delta t/m=D\Delta t$. Thus another way of looking at the equations derived is to think of wave motion as arising from statistical spreading in analogy to the sending out of waves from every point on the wave front in Huygen's principle.

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[6] R. P. Feynman, in *Selected Papers in Quantum Electrodynamics*, edited by J. Schwinger (Dover Publications, New York, 1958), p. 330.